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Isotropization in Bianchi Type IX Vacuum Cosmology

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We find the most general Bianchi type IX solution in Brans–Dicke theory (BDT) for the vacuum case, with the local rotational symmetry. For BDT coupling parameter $w > 500$ the universe becomes isotropic for any amount of initial anisotropy. In the extended inflation scenario, the Brans–Dicke scalar field ϕ can avoid the inflation in one direction.

1. INTRODUCTION

The concept of inflation in cosmology attempts to show that our present near-isotropic and near-flat universe could have been produced from arbitrary initial conditions. However, most inflationary calculations have assumed flatness and isotropy (Friedmann–Robertson–Walker, FRW, universe) to begin with; in the case of anisotropic cosmologies, most attempts have been made in the framework of general relativity theory (GRT) (for example, Futamatse *et al.*, 1989; Jensen and Stein-Schabes, 1986). Extended inflation is a new variation of inflationary theory (La and Steinhardt, 1989), based on the Brans–Dicke theory (BDT) of gravity (Brans and Dicke, 1961). There are a few solutions of extended inflation, but only for the FRW case (Duncan and Jensen, 1990).

In this paper we find, for the locally rotationally symmetric (LRS) case ($R_1 = R_2 \neq R_3$), the most general solution for the Bianchi type IX vacuum model in the BDT (there is no exact solution for the total anisotropy case, $R_1 \neq R_2 \neq R_3$); we show that (1) for large value of the BDT coupling parameter $w > 500$ [present limits based on time-delay experiments require $w \gtrsim 500$ (Reasenberg *et al.*, 1979)] the universe will become isotropic for

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any amount of initial anisotropy; there is a singularity in R_1 and R_3 in the beginning, then a phase of maximal expansion, followed finally by a contraction. (2) For small values of the temporal parameter ($\phi << 1$) we can find inflationary expansion only for one scale factor (R_3); for the other scale factor (R_1) the BDT scalar field ϕ avoids the inflationary expansion, which is in accordance with the fact that only for the $k = 0$ isotropic case is there inflation; Duncan and Jensen (1990) find inflation for $k = \pm 1$, but for radiation, not for the vacuum.

2. BIANCHI TYPE IX VACUUM FIELD EQUATIONS

The Bianchi type IX BDT vacuum field equations to be solved are

$$\dot{H}_i + 3HH_i + (\ln \phi)^{\cdot} H_i = -\frac{1}{2}(R_i^4 - R_k^4 - R_l^4 + 2R_k^2 R_l^2)R^{-6} \tag{1}$$

$$H_1H_2 + H_1H_3 + H_2H_3 - \frac{w}{2}(\ln \phi)^{\cdot 2} + 3H(\ln \phi)^{\cdot}$$
$$= \frac{1}{4}[R_1^4 + R_2^4 + R_3^4 - 2(R_2^2R_3^2 + R_1^2R_3^2 + R_1^2R_2^2)]R^{-6} \tag{2}$$

$$(R^3\dot{\phi})^{\cdot} = 0, \qquad (\ )^{\cdot} = d/dt \tag{3}$$

where $H_i = \dot{R}_i/R_i$ $(i = 1, 2, 3)$ are the Hubble parameters, $R^3 = R_1R_2R_3$, R_i are the scale factors, $\phi = \phi(t)$ is the BDT scalar field, w is the coupling parameter from the BDT, and i, k, l are in cyclic order. We consider the locally rotationally symmetric case (LRS) $R_1 = R_2$. By rescaling the proper time with the scalar field $d\phi = (1/R^3)\,dt$ we obtain equations (1)–(3) in the form

$$H_1' + \frac{1}{\phi}H_1 = \frac{1}{2}R_3^4 - R_1^2R_3^2 \tag{4}$$

$$H_3' + \frac{1}{\phi}H_3 = -\frac{1}{2}R_3^4 \tag{5}$$

$$H_1^2 + 2H_1H_3 - \frac{w}{2\phi^2} + \frac{1}{\phi}(2H_1 + H_3) = \frac{1}{4}(R_3^4 - 4R_1^2R_3^2) \tag{6}$$

where $(\cdot)' = d/d\phi$ and now $H_i = R_i'/R_i$. From equation (5) we have

$$\phi\left[\phi\,\frac{(\phi R_3^2)'}{\phi R_3^2}\right]' = -R_3^4\phi^2 \tag{7}$$

By introducing the new variables $r_3 = R_3^2\phi$ and $d\eta = d\phi/\phi$, we obtain equation (7) as

$$(\ln r_3)^{**} = -r_3^2 \tag{8}$$

where $(\cdot)^* = d/d\eta$. Equation (8) can be integrated and we obtain

$$r_3 = 4c/(4c\phi^{\sqrt{c}} + \phi^{-\sqrt{c}}) \tag{9}$$

By defining $r_{13} = R_1R_3\phi$ and following the same procedure, we obtain

$$r_{13} = 4\alpha/(4\alpha\phi^{\sqrt{\alpha}} + \phi^{-\sqrt{\alpha}}) \tag{10}$$

From the definitions of r_3 and r_{13} finally we obtain the cosmic scale factors

$$R_3 = 2c^{1/2}/\{\phi^{1/2}[4c\phi^{\sqrt{c}} + \phi^{-\sqrt{c}}]^{1/2}\} \tag{11}$$

$$R_1 = (2\alpha/c^{1/2})[4c\phi^{\sqrt{c}-1} + \phi^{-\sqrt{c}-1}]^{1/2}/[4\alpha\phi^{\sqrt{\alpha}} + \phi^{-\sqrt{\alpha}}] \tag{12}$$

with the corresponding Hubble parameters

$$H_3 = -1/(2\phi) - \frac{1}{2}\,c^{1/2}\,[4c\phi^{\sqrt{c}-1} - \phi^{-\sqrt{c}-1}]/[4c\phi^{\sqrt{c}} + \phi^{-\sqrt{c}}] \tag{13}$$

$$H_1 = 1/(2\phi) + \frac{1}{2}\,c^{1/2}\,[4c\phi^{\sqrt{c}-1} - \phi^{-\sqrt{c}-1}]/[4c\phi^{\sqrt{c}} + \phi^{-\sqrt{c}}] - \alpha^{1/2}\,[4\alpha\phi^{\sqrt{\alpha}-1} - \phi^{-\sqrt{\alpha}-1}]/[4\alpha\phi^{\sqrt{\alpha}} + \phi^{-\sqrt{\alpha}}] \tag{14}$$

By substitution of (11) and (12) into (6), we obtain α and c,

$$c = 4\alpha - (3 + 2w) \tag{15}$$

3. DISCUSSION AND CONCLUSIONS

In order to see the influence of the anisotropy on the dynamics of the universe we can define an anisotropy parameter $B = (\alpha/c)^{1/2}$.

3.1. Anisotropy

The BDT is consistent with local observations in the solar system as long as the coupling parameter w is about equal to or greater than 500 (Reasenberg *et al.*, 1979). From equation (15) for $c > 0$ this implies $\alpha > 250$. Plotting R_1 and R_3 [equations (11) and (12)] vs. the "temporal" parameter

ϕ (Fig. 1), we can see that for $\phi \to 0$ we have an initial singularity followed by a phase maximal expansion for R_1 and R_3 and for large ϕ ($\phi \to \infty$) finally we have a final singularity in a closed universe from only one cycle, which is characteristic for closed universes.

For Hubble parameters H_1 and H_3 [equations (13) and (14) and Fig. 2] we can see that for any amount of initial anisotropy, $\alpha \neq c$, the universe tends to be isotropized very rapidly ($H_1 = H_3$ when $\phi \to \infty$), but, for example, for large $w < 0$, there is no isotropy when $\phi \to \infty$ ($H_1 \neq H_3$) and the anisotropy will be greater.

3.2. B = 1. Isotropy

For this case $\sqrt{c} = \sqrt{\alpha}$ and from equations (11) and (12) we can see that $R_1 = R_3$; our Bianchi type IX solution reduces to the $k = 1$ FRW isotropic solution obtained by Cerveró and Estévez (1983).

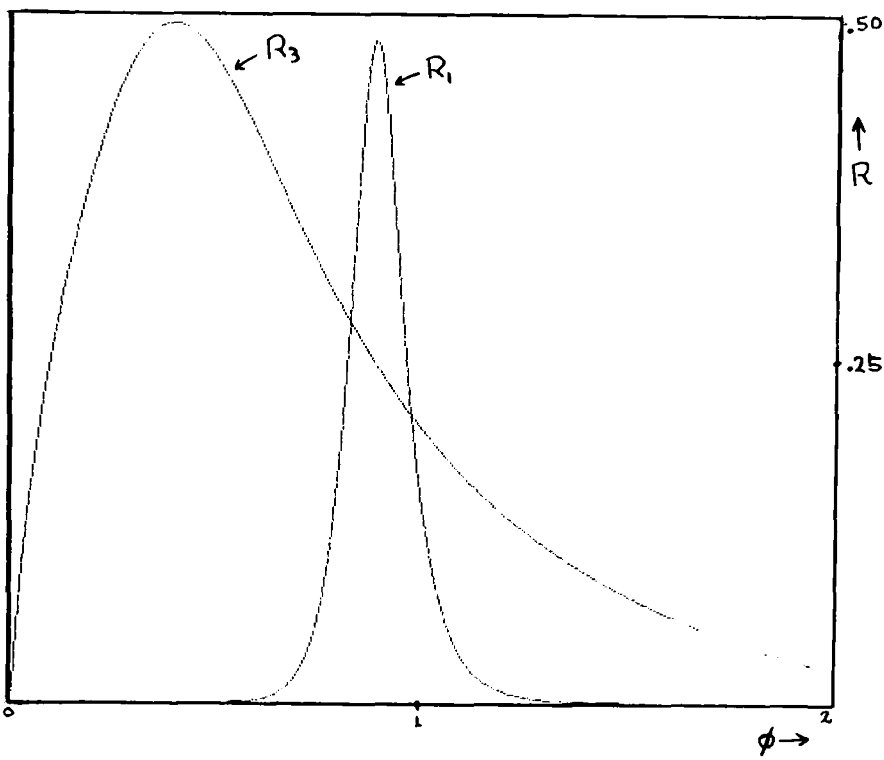

Fig. 1. "Time" dependence of the scale factors R_1 and R_3 vs. the scalar field ϕ. Here R_1 is in units of $10/2\sqrt{c}$ and R_3 in units of $2\sqrt{c}$ ($c = 2$, $\alpha = 16$).

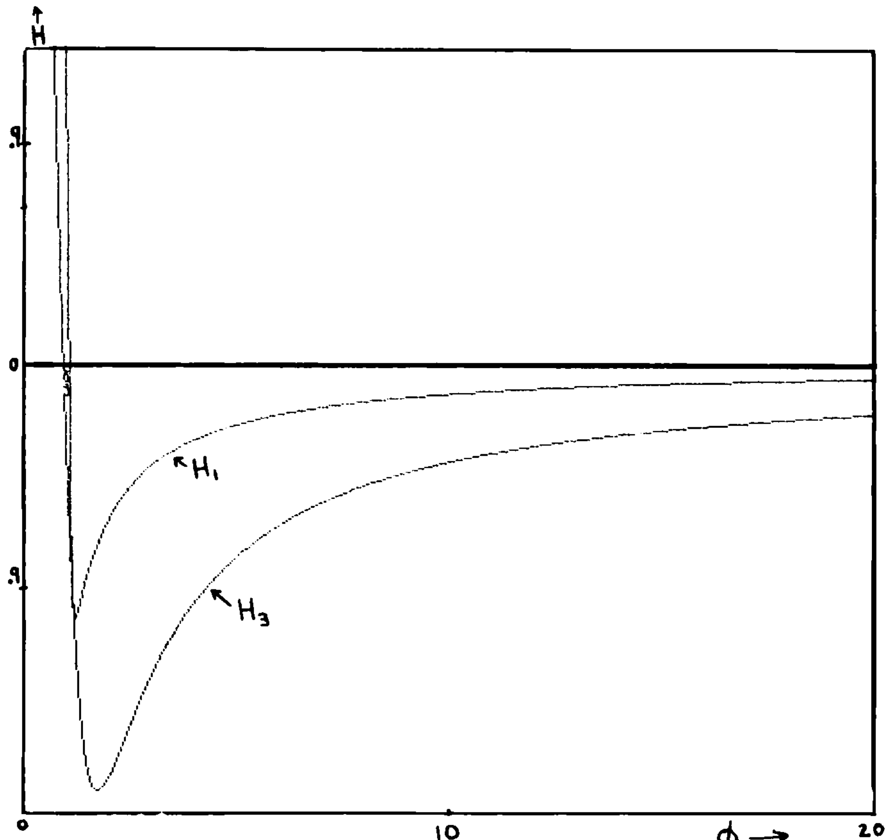

Fig. 2. "Time" dependence of the Hubble parameters H_1 and H_3 vs. ϕ. For $\phi > 20$ the universe tends to be isotropized, $H_1 \sim H_3$ (H_1 is not on the same scale as H_3).

3.3. Inflation

In order to discuss inflation in this model, we take the relationship between the proper time and the BDT scalar field ϕ given by $dt = R^3 d\phi$ and from equations (11) and (12) for $\phi << 1$ (near the initial singularity) we obtain

$$\phi \sim e^t \tag{16}$$

where the constants α and c are given by $\sqrt{c} + 1 = 4\sqrt{\alpha}$ and equation (15) for this case is valid, and we have for the scale factors of the universe in terms of the proper time t

$$R_3 \simeq e^{(\sqrt{c}/2)t}/\phi \tag{17}$$

$$R_1 \sim e^{t/2}/\phi \tag{18}$$

For $\sqrt{c} > 2$ we find inflation only for R_3; the condition for isotropization e^{60} (La and Steinhardt, 1989) is easily obtained; however, for R_1 equation (16) for the BDT scalar field ϕ avoids the inflationary expansion. For the isotropic case ($B = 1$, $\sqrt{\alpha} = \sqrt{c} = 1/3$) we have that $R^3_{\text{isotropic}} = R_1^2 R_3$,

$$R^3_{\text{isotropic}} \sim \frac{e^{t/2}}{\phi^3} \tag{19}$$

We do not have inflationary expansion; this corresponds to the FRW $k = +1$ closed universe.

In conclusion, we have found the general vacuum solution for Bianchi type IX cosmology in BDT for the LRS case, and have shown that for $w > 500$ the universe tends to be isotropized and that only for one scale factor is there inflationary behavior. In future papers we will discuss other Bianchi models in BDT.

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